Analytic formula for leading-order nonlinear coherent response in stochastic resonance

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(Received 13 July 2009; revised manuscript received 1 March 2010; published 30 March 2010)

The response of an overdamped bistable system driven by a Gaussian white noise and perturbed by a weak monochromatic signal is studied analytically. The perturbation theory is employed to calculate the nonlinear coherent response in the leading order of the amplitude of the weak signal. Simple analytic formulas for the linear and the nonlinear responses have been derived in low noise and low-frequency regime and the results based on the derived formulas are compared with the numerical results.

DOI: 10.1103/PhysRevE.81.031132 PACS number(s): 05.40.−a, 02.50.−r

I. INTRODUCTION

The stochastic resonance (SR) characterizes a cooperative phenomenon that is manifest in nonlinear systems whereby generally feeble input (a weak signal) can be amplified and optimized by the assistance of noise. It is a general belief that communication devices are disturbed by any source of background noise. Under certain circumstances, however, an extra dose of noise can in fact help rather than hinder the performance of some devices. Specifically, the output signal can be greatly enhanced (while the output noise can be considerably lessened) by suitably increasing the input noise. Because of this reason the phenomenon of SR has been found to be of relevance in signal information and detection and in a great variety of phenomenon in physics, chemistry, and the life sciences [1].

In this paper we study the response of an overdamped bistable system (nonlinear system) driven by a weak monochromatic signal and a white Gaussian noise. The response is usually defined as a switching event that carries the system from the neighborhood of one stable state to another. In the absence of periodic modulation, such switching events are purely random. However, in the presence of the modulation they become more or less correlated with it. At both low and high intensities of the external noise, the modulation and the switching events are not well correlated, but at some intermediate value they become better correlated. Given the three features of (i) nonlinearity, (ii) a weak signal, and (iii) a source of noise, the response of the system displays a nonmonotonic bell-shaped behavior with a maximum as a function of increasing noise intensity, hence the term SR.

The whole system exhibiting SR can be thought of as a signal processing device where at the input we have a periodic signal and a white noise. At the output the periodic component with the same frequency of the input signal is identified with the coherent response. The response is usually characterized by the power associated with the output. Typical quantifiers for SR are the spectral power amplification and/or the signal-to-noise ratio (SNR) [1–4]. While spectral power amplification measures the amplification of the "coherent" (periodic) power of the output over the input power contained in the periodic modulation, the SNR measures the quality of the output signal, in terms of the ratio of its " coherent" (periodic) component over its " incoherent" (noisy) component.

Here we consider a symmetric bistable potential. The noise may be inherent in the system or that may be added to the coherent deterministic periodic monochromatic signal. The framework to solve this problem is the Langevin equation. By suitably redefining the variables it is possible to characterize the system in terms of three parameters, namely, the frequency of the monochromatic periodic signal, the strength of the noise, and the amplitude of the signal.

As signal is weak, it is legitimate to analyze the system in terms of perturbation theory with the amplitude of the signal acting as a small parameter. The response of the nonlinear system is thus calculated. Linear-response theory amounts to keeping the first leading term in the perturbation expansion of the quantities in powers of the amplitude of the periodic force. It has been possible to obtain approximate analytical expressions [1,2,5] of the linear responses and the corresponding SR quantifiers for small driving force frequency and for low noise.

The recent works [4,6], however, demonstrate the importance of the nonlinear response. In this context the SR responses have been calculated by solving the Langevin equation numerically without having a recourse to the perturbation theory.

For a weak input signal the leading-order nonlinear response using perturbation theory has been put forward analytically [7] some times before. The resulting expression rightly contains several summations involving different time scales of the process. One, however, wonders whether these summations can be carried out, even approximately, in some parameter regime to express the leading-order nonlinear response in simple and compact form. This would be useful in the following ways: first, this would give first hand quick estimate of the response in the specific parameter regime. Second, the analytical formula clearly demonstrates the explicit interplay between three important parameters, namely, the frequency of the input driving, the transition rates from one well to another of the bistable potential which of course depends on the strength of the noise, and the amplitude of the input signal, exhibiting their way of cooperation or noncooperation to enhance or decrease the value of the response over its linear-response counterpart.

In what follows, after stating the problem in Sec. II, the perturbation theory is systematically developed in Sec. III

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and explicit expression of the coherent contribution of the response, which has been outlined in Sec. II, is derived in terms of the eigenvalues (related to time scales of the process) and the eigenfunctions of the unperturbed Fokker-Planck operator. In Sec. IV we will show that it has been possible to obtain simple closed-form analytical expressions for the linear and the leading-order nonlinear responses without having a detailed explicit knowledge of several time scales in low noise and low-frequency regime. It is also shown that these simple analytic expressions go over to the corresponding adiabatic expressions when the frequency of the external signal is small in comparison to all other typical frequencies of the system. This derivation involves several results that have been proved in few appendixes. These derived results are then compared with the numerical results. Finally, some concluding remarks are added in Sec. V.

II. STATEMENT OF THE PROBLEM

The Langevin equation describing the overdamped Brownian motion of a particle in a bistable potential $V(x)$, driven by a Gaussian white noise and perturbed by a weak monochromatic force $A_0 \cos \Omega t$ (the input signal), is given by

$$\dot{x} = - V'(x) + A_0 \cos \Omega t + \Gamma(t). \tag{2.1}$$

The bistable potential $V(x)$ used in Eq. (2.1) is

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4, \tag{2.2}$$

and the input noise $\Gamma(t)$ is assumed to be a Gaussian random process with zero mean and δ correlation:

$$\langle \Gamma(t) \rangle = 0, \quad \langle \Gamma(t)\Gamma(t') \rangle = 2D\delta(t-t'), \tag{2.3}$$

where the quantity D is the strength of the noise (the Gaussian random process). In Eqs. (2.1)–(2.3) all quantities are dimensionless [1] and they are especially convenient for further investigations. Henceforth we shall be using these dimensionless variables.

Corresponding to the Langevin Eqs. (2.1)–(2.3), the Fokker-Planck equation describing the time evolution of the probability distribution $P(x,t)$ reads

$$\left[\hat{L}(x,t) - \frac{\partial}{\partial t}\right]P(x,t) \equiv \left[\hat{L}_0 - A_0 \cos \Omega t \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right]P(x,t) = 0, \tag{2.4}$$

where the unperturbed operator $\hat{L}_0$ refers to the unperturbed $(A_0=0)$ process:

$$\hat{L}_0(x) = \frac{\partial}{\partial x}V'(x) + D\frac{\partial^2}{\partial x^2}. \tag{2.5}$$

The symmetric bistable potential Eq. (2.2) has a barrier height $\Delta V=\frac{1}{4}$ and two minima and one maximum at $x=\pm 1$ and $x=0$, respectively. As $\hat{L}(x,t)=\hat{L}(x,t+T)$, with $T=(2\pi/\Omega)$, Eq. (2.4) admits Floquet solution of the form

$$P(x,t) = e^{-\mu t}p_\mu(x,t), \tag{2.6}$$

with μ being the Floquet eigenvalue and $p_\mu(x,t)$ being the periodic Floquet eigenfunction:

$$p_\mu(x,t) = p_\mu(x,t+T). \tag{2.7}$$

Substituting the ansatz Eq. (2.6) into Eq. (2.4) one obtains the eigenvalue equation satisfied by the Floquet mode $p_\mu(x,t)$:

$$\left[\hat{L}(x,t) - \frac{\partial}{\partial t}\right]p_\mu(x,t) = - \mu p_\mu(x,t). \tag{2.8}$$

Here the Floquet modes $\{p_\mu(x,t)\}$ belong to the product space $L_1(X)\otimes T_\Omega$, where T_Ω is the space of functions that are periodic in time with period T and $L_1(X)$ is the linear space of integrable functions [1]. The basis in T_Ω may be chosen as $\{e^{in\Omega t}\}_{n=-\infty}^{\infty}$ and the asymptotic periodic solution, $p_0(x,t)$ admits an expansion [8]:

$$p_0(x,t) = \sum_{n=-\infty}^{\infty} C_n(x;\mu=0)e^{in\Omega t}. \tag{2.9}$$

It is to be noted that as the asymptotic solution $p_0(x,t)$ is real, $C_n^*(x)=C_{-n}(x)$. The coherent part of the one-time correlation function, $C_{coh}(\tau)$ that does not involve any exponential damping in the asymptotic correlation $\langle X(t+\tau)X(t)\rangle_{asy}$ is [4]

$$C(\tau) = \frac{1}{T}\int_0^T dt\langle X(t+\tau)X(t)\rangle_{asy} = C_{coh}(\tau) + C_{incoh}(\tau),$$

$$C_{coh}(\tau) = \frac{1}{T}\int_0^T dt\langle X(t)\rangle_{asy}\langle X(t+\tau)\rangle_{asy}, \tag{2.10}$$

where $\langle X(t)\rangle_{asy}$ is the asymptotic mean of the stochastic position variable $X(t)$. The Fourier transform of the one-time correlation function is defined as

$$\tilde{C}_{coh}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} C_{coh}(\tau)e^{i\omega\tau}d\tau. \tag{2.11}$$

Now, $\bar{C}_{coh}(\Omega,D,A_0)$ corresponds to the component of $\tilde{C}_{coh}(\omega,D,A_0)$ at the frequency $\omega=\Omega$ [4]:

$$\bar{C}_{coh}(\Omega,D,A_0) = \lim_{\epsilon\to 0}\int_{\omega=\Omega-\epsilon}^{\omega=\Omega+\epsilon} d\omega\tilde{C}_{coh}(\omega,D,A_0). \tag{2.12}$$

It is then easy to see that

$$\bar{C}_{coh}(\Omega,D,A_0) = 2\left|\int_{-\infty}^{\infty} xC_1(x;\mu=0)dx\right|^2. \tag{2.13}$$

The coherent response is a measure of the power of the signal at the output. The power amplification at the frequency $\omega=\Omega$, introduced as a quantifier of SR [1,2], relates to the ratio of the signal power at the output over that at the input and it is given by

$$\eta(\Omega,D,A_0) = \frac{2\bar{C}_{coh}(\Omega,D,A_0)}{A_0^2}. \tag{2.14}$$

In what follows we will derive $\bar{C}_{coh}(\Omega,D,A_0)$ and $\eta(\Omega,D,A_0)$ taking into account both linear and nonlinear responses.

III. RESPONSE THROUGH PERTURBATION

Bistability is generally accepted to be the key ingredient of SR. In the present paper we explore the response due to the periodic perturbation $A_0 \cos \Omega t$. For this weak external forcing, we apply the perturbation theory with the amplitude A_0 acting as a small parameter. Our aim is to evaluate $\bar{C}_{coh}(\Omega,D,A_0)$ in Eq. (2.13) up to the leading-order nonlinear response. In order to evaluate this quantity one needs to know the Floquet eigenfunction $p_0(x,t)$. Here, we will develop the perturbation theory for the full set of $\{p_{\mu_l}(x,t)\}$ and the corresponding Floquet eigenvalues $\{\mu_l\}$ for convenience and for the sake of generality. We will derive these quantities in terms of the eigenfunctions $\{\phi_l(x)\}$ and the eigenvalues $\{\lambda_l\}$ of the unperturbed Fokker-Planck operator $\hat{L}_0$ introduced in Eq. (2.5);

$$\hat{L}_0\phi_l(x) = -\lambda_l\phi_l(x). \tag{3.1}$$

Here, the potential $V(x)$ diverges very fast as $|x|^\alpha$ with $\alpha > 1$, as $|x|\to\infty$. This ensures that Eq. (3.1) supports a stationary solution, $\phi_0(x)$ with $\lambda_0=0$ and $\lambda_l \geq 0$, $\forall\ l$. We arrange them: $0=\lambda_0<\lambda_1<\lambda_2<\lambda_3<\cdots$. For this symmetric potential Eq. (2.2), $\phi_l(x)=(-1)^l\phi_l(-x)$, $\forall\ l$. The eigenfunctions satisfy the normalization condition:

$$\langle\phi_m^\dagger|\phi_l\rangle = \int \phi_m^\dagger(x)\phi_l(x)dx = \delta_{m,l}, \tag{3.2}$$

where $\{\phi_m^\dagger(x)\}$ are the eigenfunctions of the operator adjoint to $\hat{L}_0$ and are given by

$$\hat{L}_0^\dagger\phi_l^\dagger(x) = -\lambda_l\phi_l^\dagger(x), \quad \phi_l^\dagger = \frac{\phi_l}{\phi_0}, \quad \forall\ l. \tag{3.3}$$

Equation (3.3) implies $\phi_0^\dagger=1$. The sets $\{\phi_l(x)\}$ and $\{\lambda_l\}$ are assumed to be known.

The perturbation theory that we will be discussing in this paper is different from the approach adopted in [7]. In [7] the Fokker-Planck Eq. (2.4) was transformed into the Schrodinger form where the resulting operator is Hermitian in $L_1(X)$ and the corresponding propagator was expanded in powers perturbation with the help of the propagator of the unperturbed Hermitian operator, according to the perturbation formula due to Feynman [9]. The unperturbed propagator was also then expanded in terms of the eigenfunctions to express the one-time correlation function $C(\tau)$ in terms of different matrix elements. The perturbation theory that we will consider here starts directly from the original Fokker-Planck equation where the operator $\hat{L}(x,t)$ is not Hermitian in $L_1(X)$. This perturbation theory is sketched in the Appendix of [1]. Here we develop it systematically to calculate the perturbed quantities to any arbitrary order of the amplitude of the monochromatic signal. We start with the Floquet eigenvalue Eqs. (2.8) with $\hat{L}(x,t)$ defined in Eq. (2.4). Expanding the Floquet eigenfunctions in terms of the Fourier basis $\{e^{in\Omega t}\}_{n=-\infty}^{\infty}$,

$$p_\mu(x,t) = \sum_{n=-\infty}^{\infty} C_n(x;\mu)e^{in\Omega t}, \tag{3.4}$$

and substituting in Eq. (2.8), we obtain the hierarchy of coupled ordinary differential equations:

$$[\hat{L}_0 - in\Omega + \mu]C_n(x) = \frac{A_0}{2}[C'_{n-1}(x) + C'_{n+1}(x)]. \tag{3.5}$$

where $C'_n(x)=\frac{dC_n(x)}{dx}$. We now seek a solution of Eq. (3.5) in terms of the perturbation expansion over the unperturbed states $\{\phi_l(x)\}$:

$$C_n(x;\mu_l) = \phi_l\delta_{n,0} + \sum_{k=1}^{\infty} A_0^k C_n^{(k)}(x;\mu_l); \quad n=0, \pm 1, \pm 2,\ldots \tag{3.6}$$

In the above expansion nonzero values of subscript n correspond to nonzero values of frequencies (fundamental and higher harmonics), the superscript k refers to the kth-order perturbation and $k=0$ term corresponds to the unperturbed state $\phi_l(x)$. Similarly, the perturbation expansion for the Floquet eigenvalue:

$$\mu_l = \mu_l^{(0)} + \sum_{k=1}^{\infty} A_0^k\mu_l^{(k)}. \tag{3.7}$$

Substituting Eqs. (3.6) and (3.7) into Eq. (3.5) for $\mu=\mu_l$ and equating the coefficients of different powers of A_0 to zero, we obtain

$$[\hat{L}_0 + \mu_l^{(0)}]\phi_l(x) = 0, \tag{3.8}$$

$$\delta_{n,0}\phi_l\mu_l^{(1)} + [\hat{L}_0 - in\Omega + \mu_l^{(0)}]C_n^{(1)}(x;\mu_l) - \frac{1}{2}\phi_l'(x)[\delta_{n,-1} + \delta_{n,1}] = 0, \tag{3.9}$$

$$\delta_{n,0}\phi_l\mu_l^{(k)} + [\hat{L}_0 - in\Omega + \mu_l^{(0)}]C_n^{(k)}(x;\mu_l) + \sum_{r=1}^{k-1}\mu_l^{(r)}C_n^{(k-r)}(x;\mu_l) - \frac{1}{2}[C_{n-1}^{(k-1)\prime}(x;\mu_l) + C_{n+1}^{(k-1)\prime}(x;\mu_l)] = 0; \quad k \geq 2. \tag{3.10}$$

Equation (3.1) suggests that

$$\mu_l^{(0)} = \lambda_l. \tag{3.11}$$

Taking inner product of Eq. (3.9) with $\phi_m^\dagger(x)$ we obtain

$$\delta_{n,0}\delta_{m,l}\mu_l^{(1)} + (\lambda_l - \lambda_m - in\Omega)\langle\phi_m^\dagger|C_n^{(1)}(\mu_l)\rangle - \frac{1}{2}\langle\phi_m^\dagger|\frac{\partial}{\partial x}|\phi_l\rangle[\delta_{n,-1} + \delta_{n,1}] = 0, \tag{3.12}$$

where use has been made of the orthonormality condition Eq. (3.2), the eigenvalue Eq. (3.3) for the unperturbed adjoint operator $\hat{L}_0^\dagger(x)$ and Eq. (3.11). With $n=0$, $m=l$, Eq.

(3.12) immediately gives the first order perturbation of the Floquet eigenvalue:

$$\mu_l^{(1)} = 0. \tag{3.13}$$

Employing Eq. (3.13), Eq. (3.12) reduces to

$$(\lambda_l - \lambda_m - in\Omega)\langle \phi_m^\dagger | C_n^{(1)}(\mu_l)\rangle - \frac{1}{2}\langle \phi_m^\dagger | \frac{\partial}{\partial x} | \phi_l\rangle [\delta_{n,-1} + \delta_{n,1}] = 0. \tag{3.14}$$

Taking $n=0$, Eq. (3.14) yields

$$\langle \phi_m^\dagger | C_0^{(1)}(\mu_l)\rangle = 0; \;\; m \neq l, \tag{3.15}$$

with $\langle \phi_l^\dagger | C_0^{(1)}(\mu_l)\rangle$ being an arbitrary constant. Without loss of generality we take it to be zero. Therefore we have

$$C_0^{(1)}(x;\mu_l) = 0. \tag{3.16}$$

Taking $n= \pm 1$, Eq. (3.14) gives

$$\langle \phi_m^\dagger | C_{\pm 1}^{(1)}(\mu_l)\rangle = \frac{\langle \phi_m^\dagger | \frac{\partial}{\partial x} | \phi_l\rangle}{2(\lambda_l - \lambda_m \mp i\Omega)}, \quad \forall \;\; m, \tag{3.17}$$

while for $n \neq 0, \pm 1$, this provides

$$C_n^{(1)}(x;\mu_l) = 0, \quad n \neq \pm 1, \tag{3.18}$$

since $\{\phi_m^\dagger(x)\}$ form a complete set. Next, we take inner product of Eq. (3.10) with $\phi_m^\dagger(x)$ and use the orthonomality condition (3.2) to obtain

$$\delta_{n,0}\delta_{m,l}\mu_l^{(k)} + (\lambda_l - \lambda_m - in\Omega)\langle \phi_m^\dagger | C_n^{(k)}(\mu_l)\rangle + \sum_{r=1}^{k-1} \mu_l^{(r)} \times \langle \phi_m^\dagger | C_n^{(k-r)}(\mu_l)\rangle - \frac{1}{2}\langle \phi_m^\dagger | \frac{\partial}{\partial x} | [C_{n-1}^{(k-1)}(\mu_l) + C_{n+1}^{(k-1)}(\mu_l)]\rangle = 0, \quad k \geq 2. \tag{3.19}$$

Substituting $n=0$, $m=l$ in Eq. (3.19) we obtain the recurrence relation of different perturbation orders of Floquet eigenvalue μ_l:

$$\mu_l^{(k)} = - \sum_{r=1}^{k-1} \mu_l^{(r)} \langle \phi_l^\dagger | C_0^{(k-r)}(\mu_l)\rangle + \frac{1}{2}\langle \phi_l^\dagger | \frac{\partial}{\partial x} | [C_{-1}^{(k-1)}(\mu_l) + C_1^{(k-1)}(\mu_l)]\rangle, \quad k \geq 2. \tag{3.20}$$

Further, taking $n \neq 0$, one gets readily from Eq. (3.19):

$$\langle \phi_m^\dagger | C_n^{(k)}(\mu_l)\rangle = \frac{1}{(\lambda_l - \lambda_m - in\Omega)} \left[- \sum_{r=1}^{k-1} \mu_l^{(r)} \langle \phi_m^\dagger | C_n^{(k-r)}(\mu_l)\rangle + \frac{1}{2}\langle \phi_m^\dagger | \frac{\partial}{\partial x} | [C_{n-1}^{(k-1)}(\mu_l) + C_{n+1}^{(k-1)}(\mu_l)]\rangle \right], \quad \forall \;\; m, \;\; k \geq 2, \tag{3.21}$$

while for $n=0$ one recovers:

$$\langle \phi_m^\dagger | C_0^{(k)}(\mu_l)\rangle = \frac{1}{(\lambda_l - \lambda_m)} \left[- \sum_{r=1}^{k-1} \mu_l^{(r)} \langle \phi_m^\dagger | C_0^{(k-r)}(\mu_l)\rangle + \frac{1}{2}\langle \phi_m^\dagger | \frac{\partial}{\partial x} | [C_{-1}^{(k-1)}(\mu_l) + C_1^{(k-1)}(\mu_l)]\rangle \right], \quad m \neq l, \;\; k \geq 2, \tag{3.22}$$

$$\langle \phi_l^\dagger | C_0^{(k)}(\mu_l)\rangle = \text{arbitrary constants}, \quad k \geq 2. \tag{3.23}$$

Equations (3.20)–(3.23) constitute the perturbations of Floquet's eigenvalues and eigenfunctions to an arbitrary order.

Our interest is to calculate $\bar{C}_{coh}(\Omega,D,A_0)$ in Eq. (2.13) and the spectral power amplification, $\eta(\Omega,D,A_0)$ which is related to $\bar{C}_{coh}(\Omega,D,A_0)$ by Eq. (2.14). Using the above perturbation theory one immediately has

$$\int_{-\infty}^{\infty} x C_1(x;\mu=0)dx = A_0 \langle \phi_0^\dagger | x | C_1^{(1)}(\mu=0)\rangle + A_0^3 \langle \phi_0^\dagger | x | C_1^{(3)}(\mu=0)\rangle + O(A_0^4), \tag{3.24}$$

where we have used the perturbation result $C_n^{(2)}(x;\mu_l) = 0$, $n \neq 0, \pm 2$. Now, the linear-response theory amounts to keeping the first leading term in the perturbation expansion of the quantities in powers of the amplitude of the periodic force $A_0 \cos \Omega t$. We use the superscript (L) to the quantities taken in the linear-response theory. The higher-order terms correspond to the nonlinear response. We will calculate explicitly the leading-order nonlinear response which will be denoted with a superscript (NL). Thus we write

$$\bar{C}_{coh}(\Omega,D,A_0) = \bar{C}_{coh}^{(L)}(\Omega,D,A_0) + \bar{C}_{coh}^{(NL)}(\Omega,D,A_0). \tag{3.25}$$

Employing completeness relation $\Sigma_m |\phi_m\rangle\langle\phi_m^\dagger| = 1$ and the matrix elements derived in the above perturbation theory, $\bar{C}_{coh}^{(L)}(\Omega,D,A_0)$ and $\bar{C}_{coh}^{(NL)}(\Omega,D,A_0)$ are expressed in terms of the eigenfunctions and eigenvalues of the unperturbed Fokker-Planck operator $\hat{L}_0(x)$:

$$\bar{C}_{coh}^{(L)}(\Omega,D,A_0) = \frac{1}{2}A_0^2 |\chi_1(\Omega,D)|^2, \tag{3.26}$$

$$\bar{C}_{coh}^{(NL)}(\Omega,D,A_0) = A_0^4[\chi_1(\Omega,D)(XI) + \text{c.c.}], \tag{3.27}$$

$$(XI) = - \frac{1}{2^3} \sum_m \frac{\langle \phi_0^\dagger | x | \phi_m\rangle}{(\lambda_m - i\Omega)} \sum_r a_{mr} \left[\frac{1}{(\lambda_r - 2i\Omega)} \sum_s \frac{a_{rs}a_{s0}}{(\lambda_s - i\Omega)} + \frac{1}{\lambda_r} \sum_s a_{rs}a_{s0} \left(\frac{1}{(\lambda_s - i\Omega)} + \frac{1}{(\lambda_s + i\Omega)} \right) \right], \tag{3.28}$$

where the linear-response function $\chi_{\pm 1}(\Omega,D)$ is defined as

$$\chi_{\pm 1}(\Omega,D) = 2\langle \phi_0^\dagger | x | C_{\pm 1}^{(1)}(\mu=0)\rangle = -\sum_m \frac{\langle \phi_0^\dagger | x | \phi_m\rangle a_{m0}}{\lambda_m \pm i\Omega}, \tag{3.29}$$

the quantity a_{lr} is given by

$$a_{lr} = \langle \phi_l^\dagger | \frac{\partial}{\partial x} | \phi_r\rangle, \tag{3.30}$$

and the notation c.c. in Eq. (3.27) indicates the complex conjugate of the bracketed portion associated with it.

The corresponding components of the signal amplification factor will be obviously given by

$$\eta^{(L)/(NL)}(\Omega,D,A_0) = \frac{2\bar{C}_{coh}^{(L)/(NL)}(\Omega,D,A_0)}{A_0^2}. \tag{3.31}$$

IV. APPROXIMATE EVALUATION OF $\bar{C}_{coh}(\Omega,D,A_0)$ AND $\eta(\Omega,D,A_0)$ IN LOW-FREQUENCY LOW NOISE REGIME

In this section we attempt to get a closed analytical form of $\bar{C}_{coh}(\Omega,D,A_0)$ in Eq. (3.25). As has been argued in the introduction, if it is possible in some parameter regime it would provide a first hand quick estimate of the response. Explicit evaluation of $\bar{C}_{coh}(\Omega,D,A_0)$ for arbitrary (Ω,D,A_0) obviously requires a knowledge of the full set of eigenvalues and eigenfunctions of the unperturbed Fokker-Planck operator $\hat{L}_0(x)$. It is practically impossible to obtain a closed-form analytical expression of $\bar{C}_{coh}(\Omega,D,A_0)$ even if we know analytical expressions of the sets $\{\lambda_l\}$, $\{\phi_l(x)\}$ as a function of the noise strength, D; the summation may not be carried out analytically to obtain a neat closed form. Unfortunately, the eigenvalues $\{\lambda_l\}$ and eigenfunctions $\{\phi_l(x)\}$ for the Fokker Planck operator $\hat{L}_0(x)$ in Eq. (3.1) for the potential Eq. (2.2) as a function of D are not known explicitly to the best of our knowledge. The lowest eigenvalue, $\lambda_0=0$ and the corresponding eigenfunction $\phi_0(x)$ are known explicitly. The approximate expression of λ_1, which relates to the rate of escape from the potential well, is also known [10] explicitly for small noise strength. It is also known [5,11] that for low noise strength,

$$\lambda_1 \ll \lambda_2 < \lambda_3 < \cdots \tag{4.1}$$

$$\lambda_p \sim O(p), \quad p \geq 2. \tag{4.2}$$

In order to utilize these conditions we restrict our derivation in low noise regime. We further assume that the driving frequency of the perturbed force $A_0 \cos \Omega t$ is much less than unity such that

$$\Omega^2 < \Omega < \lambda_p^2, \quad p \geq 2. \tag{4.3}$$

We now show that in the parameter regime defined by the conditions Eqs. (4.1)–(4.3) it is possible to evaluate $\bar{C}_{coh}(\Omega,D,A_0)$ in a closed analytical form.

A. Approximate closed-form expression of $\bar{C}_{coh}^{(L)}(\Omega,D,A_0)$ and $\eta^{(L)}(\Omega,D)$

The consideration of parity of $\{\phi_m(x)\}$ suggests that the index m in the summation of linear-response function, $\chi_1(\Omega,D)$ in Eq. (3.29) survives only for odd values of m. With the help of Eq. (A7) $\chi_1(\Omega,D)$ rewrites

$$\chi_1(\Omega,D) = \frac{1}{D}\sum_{m=1}^{\infty} \frac{\lambda_m}{\lambda_m + i\Omega}\langle \phi_0^\dagger | x | \phi_m\rangle^2. \tag{4.4}$$

Next, utilizing the conditions Eqs. (4.2) and (4.3) we have

$$\frac{1}{\lambda_m + i\Omega} = \frac{\lambda_m}{\lambda_m^2 + \Omega^2} - i\frac{\Omega}{\lambda_m^2 + \Omega^2} \simeq \frac{1}{\lambda_m}, \quad m \geq 2. \tag{4.5}$$

Thus the expression of the linear-response function in this parameter regime simplifies to

$$\chi_1(\Omega,D) \approx \frac{1}{D}\left[\frac{\lambda_1}{\lambda_1 + i\Omega}\langle \phi_0^\dagger | x | \phi_1\rangle^2 + \nu(D)\right], \tag{4.6}$$

where the quantity $\nu(D)$ is defined as

$$\nu(D) = \sum_{m=3}^{\infty}\langle \phi_0^\dagger | x | \phi_m\rangle^2 = \sum_{m=1}^{\infty}\langle \phi_0^\dagger | x | \phi_m\rangle^2 - \langle \phi_0^\dagger | x | \phi_1\rangle^2 = \langle \phi_0^\dagger | x^2 | \phi_0\rangle - \langle \phi_0^\dagger | x | \phi_1\rangle^2. \tag{4.7}$$

For low noise strength, D, $\lambda_1 = (\frac{\sqrt{2}}{\pi})e^{(-\triangle V/D)}$, $\Delta V = \frac{1}{4}$ [Eq. (2.2)]. The quantities $\langle \phi_0^\dagger | x^2 | \phi_0\rangle$ and $\langle \phi_0^\dagger | x | \phi_1\rangle^2$ are plotted in [12] from where the value of $\nu(D)$ can be read off. Its value is small in low noise strength regime. Substituting this approximate expression Eq. (4.6) into Eq. (3.26), we obtain the linear coherent response $\bar{C}_{coh}^{(L)}(\Omega,D,A_0)$ as

$$\bar{C}_{coh}^{(L)}(\Omega,D,A_0) = \alpha A_0^2, \tag{4.8}$$

where $\alpha(\Omega,D)$ is given by

$$\alpha(\Omega,D) = \frac{1}{2}|\chi_1(\Omega,D)|^2 \approx \frac{1}{2D^2}\left(\frac{\lambda_1^2}{\lambda_1^2 + \Omega^2}\right)\langle \phi_0^\dagger | x^2 | \phi_0\rangle^2. \tag{4.9}$$

In Eq. (4.9) we have used $\nu^2 \simeq 0$, as ν is a very small quantity for low D. With typical values of $D=0.2$, $\Omega=0.1$ the approximate expression Eqs. (4.8) and (4.9) of $\bar{C}_{coh}^{(L)}(\Omega,D,A_0)$ is plotted against A_0^2 (solid straight line curve) and compared with the numerical data (dotted curve) [4] in Fig. 1. It matches very well.

The signal amplification factor corresponding to the linear response, $\eta^{(L)}$ can therefore be obtained using Eqs. (3.31) and (4.8) as

$$\eta^{(L)}(\Omega,D) = 2\alpha(\Omega,D) \approx \frac{1}{D^2}\left(\frac{\lambda_1^2}{\lambda_1^2 + \Omega^2}\right)\langle \phi_0^\dagger | x^2 | \phi_0\rangle^2. \tag{4.10}$$

The approximate expression of $\eta^{(L)}(\Omega,D)$ with $\Omega=0.1$ is plotted against D (solid curve) in Fig. 2. It agrees with the result (open circles) [2].

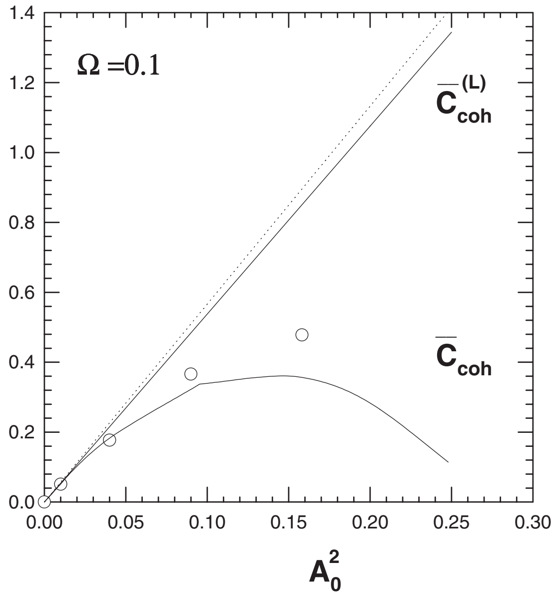

FIG. 1. $\bar{C}_{coh}^{(L)}$ and $\bar{C}_{coh}$ vs A_0^2 (solid lines), $\bar{C}_{coh}^{(L)}$ vs A_0^2 from [4] (dotted line), $\bar{C}_{coh}$ vs A_0^2 from the numerical solution of the Langevin equation (open circles) [2,4] with $\Omega=0.1$, $D=0.2$.

B. Approximate closed-form expression of $\bar{C}_{coh}(\Omega,D,A_0)$ and $\eta(\Omega,D,A_0)$

In this subsection we will show that it is possible to obtain a neat closed-form of Eqs. (3.27) and (3.28) in low-frequency low noise regime where the conditions Eqs. (4.1)–(4.3) are valid. In what follows we will show that $\bar{C}_{coh}^{(NL)}(\Omega,D,A_0)$ can be expressed neatly in a simple form:

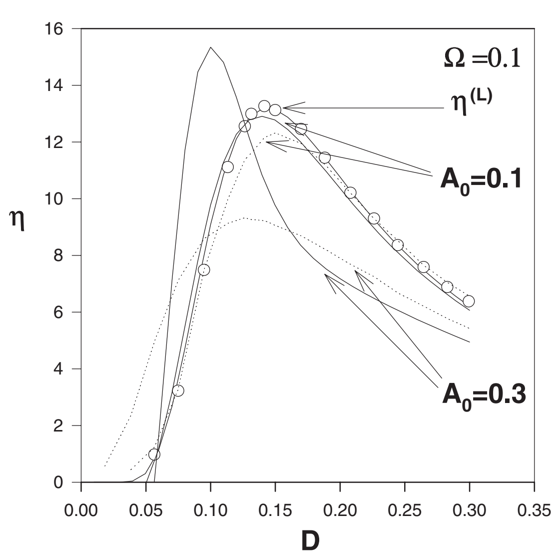

FIG. 2. $\eta^{(L)}$ vs D (solid line) and (open circles from [2]) with $\Omega=0.1$, η vs D (solid line) and from the numerical results obtained using the MCF technique (dotted line) [2] with $\Omega=0.1$, $A_0=0.1,0.3$.

$$\bar{C}_{coh}^{(NL)}(\Omega,D,A_0) \approx A_0^2\bar{C}_{coh}^{(L)}(\Omega,D,A_0)f_1(\Omega,D) + A_0^4\nu f_2(\Omega,D). \tag{4.11}$$

This derivation is based on some new results whose proofs are given in the Appendixes A–G. Here, we outline the relevant steps in arriving at the final form with frequent reference to the appropriate equations in the Appendixes A–G.

The consideration of parity of $\{\phi_m(x)\}$ and application of the conditions, Eqs. (4.2) and (4.3) simplify the expression Eq. (3.28) to

$$(XI) = (XI)_{m=1} + (XI)_{m\neq 1}, \tag{4.12}$$

$$(XI)_{m=1} = -\left(\frac{1}{2^3}\right)\frac{\langle\phi_0^\dagger|x|\phi_1\rangle}{(\lambda_1 - i\Omega)}F_{XI}(m=1), \tag{4.13}$$

$$(XI)_{m\neq 1} \approx -\left(\frac{1}{2^3}\right)\sum_{m=3}^{\infty}\frac{\langle\phi_0^\dagger|x|\phi_m\rangle}{\lambda_m}F_{XI}(m), \tag{4.14}$$

$$F_{XI}(m) = [F_{XI}(m)]_{s=1} + [F_{XI}(m)]_{s\neq 1}, \tag{4.15}$$

$$[F_{XI}(m)]_{s=1} = a_{10}\left(\frac{2}{\lambda_1 - i\Omega} + \frac{1}{\lambda_1 + i\Omega}\right)\sum_{r=2}^{\infty}\frac{a_{mr}a_{r1}}{\lambda_r}, \tag{4.16}$$

$$[F_{XI}(m)]_{s\neq 1} \approx 3\sum_{r=2}^{\infty}\frac{a_{mr}}{\lambda_r}\sum_{s=3}^{\infty}\frac{a_{rs}a_{s0}}{\lambda_s}. \tag{4.17}$$

In the derivation we first evaluate $[F_{XI}(m)]_{s\neq 1}$. With the help of Eqs. (A7), (B7), (B12), and (A9) successively the following sums can be evaluated exactly:

$$3\sum_{r=2}^{\infty}\frac{a_{mr}}{\lambda_r}\sum_{s=1}^{\infty}\frac{a_{rs}a_{s0}}{\lambda_s} = \left(\frac{3}{2D^2}\right)\sum_{r=2}^{\infty}a_{mr}\langle\phi_r^\dagger|x^2|\phi_0\rangle$$
$$= \left(\frac{1}{2D^3}\right)[C_2\lambda_m\langle\phi_m^\dagger|x|\phi_0\rangle - \lambda_m^2\langle\phi_m^\dagger|x|\phi_0\rangle], \tag{4.18}$$

where the factor C_2 is given by

$$C_2 = 3\langle\phi_0^\dagger|x^2|\phi_0\rangle - 1. \tag{4.19}$$

With this result, Eq. (4.18) $F_{XI}(m)$ in Eq. (4.15) is reexpressed as

$$F_{XI}(m) \approx \left(\frac{1}{2D^3}\right)[C_2\lambda_m\langle\phi_m^\dagger|x|\phi_0\rangle - \lambda_m^2\langle\phi_m^\dagger|x|\phi_0\rangle] + C_1 f_m, \tag{4.20}$$

where the factors C_1 and f_m are given by

$$C_1 = \frac{3\langle\phi_1^\dagger|x|\phi_0\rangle}{D} + a_{10}\left(\frac{2}{\lambda_1 - i\Omega} + \frac{1}{\lambda_1 + i\Omega}\right),$$

$$f_m = \sum_{r=2}^{\infty}\frac{a_{mr}a_{r1}}{\lambda_r}. \tag{4.21}$$

Having obtained the expression of $F_{XI}(m)$ in Eq. (4.20) we next evaluate the quantity (XI) through Eqs. (4.12)–(4.14). Employing Eqs. (G7), (4.7), (E6), and (F15) we have

$$(XI)_{m=1} \approx -\left(\frac{1}{2^4}\right)\left(\frac{1}{D^3}\right)\left(\frac{\lambda_1}{\lambda_1 - i\Omega}\right)\langle \phi_0^\dagger|x|\phi_1\rangle^2 (C_2 - \lambda_1) - \left(\frac{1}{2^3}\right)\left(\frac{1}{D^2}\right)\left(\frac{\lambda_1}{\lambda_1 - i\Omega}\right)\langle \phi_0^\dagger|x|\phi_1\rangle g_1 C_1, \quad (4.22)$$

$$(XI)_{m\neq 1} \approx -\left(\frac{1}{2^4}\right)\left(\frac{1}{D^3}\right)(-D + \lambda_1\langle \phi_0^\dagger|x|\phi_1\rangle^2 + \nu C_2) - \left(\frac{1}{2^3}\right)\left(\frac{1}{D^2}\right)\langle \phi_0^\dagger|x|\phi_1\rangle g_2 C_1, \quad (4.23)$$

where the quantities g_1 and g_2 are defined as

$$g_1 = \frac{1}{2}\langle \phi_0^\dagger|x^2|\phi_0\rangle - \langle \phi_0^\dagger|x|\phi_1\rangle^2,$$

$$g_2 = \frac{1}{2}(1+\lambda_1) - \langle \phi_0^\dagger|x^2|\phi_0\rangle - g_1. \quad (4.24)$$

Finally, substituting the approximate expressions for $\chi_1(\Omega,D)$ and (XI) from Eqs. (4.6), (4.22), and (4.23) into Eq. (3.27) we obtain the approximate closed-form of the leading-order nonlinear response, $\bar{C}_{coh}^{(NL)}(\Omega,D,A_0)$:

$$\bar{C}_{coh}^{(NL)}(\Omega,D,A_0) \approx -\left(\frac{1}{4}\right)\left(\frac{A_0}{D}\right)^2 \bar{C}_{coh}^{(L)}(\Omega,D,A_0) \times\left[f^{(ad)}(D) + 2\left(\frac{\Omega^2}{\lambda_1^2+\Omega^2}\right)fac_1(D)\right] - \left(\frac{\nu}{8}\right)\left(\frac{A_0}{D}\right)^4\left(\frac{\Omega^2}{\lambda_1^2+\Omega^2}\right)fac_2(D), \quad (4.25)$$

where $\bar{C}_{coh}^{(L)}(\Omega,D,A_0)$ is already obtained in Eqs. (4.8) and (4.9) and $f^{(ad)}(D)$, $fac_1(D)$, $fac_2(D)$ are given by

$$f^{(ad)}(D) = C_2 - \frac{D}{\langle \phi_0^\dagger|x^2|\phi_0\rangle},$$

$$fac_1(D) = 3g_1 + k_1^c, \quad fac_2(D) = \langle \phi_0^\dagger|x^2|\phi_0\rangle(l_1 + 3l_1^c),$$

$$k_1^c = \delta^2(1+\lambda_1) - \langle \phi_0^\dagger|x|\phi_1\rangle^2, \quad l_1 = \lambda_1 - \frac{D}{\langle \phi_0^\dagger|x^2|\phi_0\rangle},$$

$$l_1^c = \delta(1+\lambda_1) - \langle \phi_0^\dagger|x^2|\phi_0\rangle,$$

$$\delta = \frac{\langle \phi_0^\dagger|x|\phi_1\rangle^2}{\langle \phi_0^\dagger|x^2|\phi_0\rangle}. \quad (4.26)$$

Thus taking into account the leading-order nonlinear response, $\bar{C}_{coh}(\Omega,D,A_0)$ in Eq. (3.25) finally takes the simple approximate form:

$$\bar{C}_{coh}(\Omega,D,A_0) \approx \bar{C}_{coh}^{(L)}(\Omega,D,A_0)\left[1 - \left(\frac{A_0^2}{4D^2}\right)f^{(ad)}(D)\right] - \left(\frac{A_0^2}{2D^2}\right)\left(\frac{\Omega^2}{\lambda_1^2+\Omega^2}\right)\left[\bar{C}_{coh}^{(L)}(\Omega,D,A_0) \times fac_1(D) + \left(\frac{\nu}{4}\right)\left(\frac{A_0}{D}\right)^2 fac_2(D)\right]. \quad (4.27)$$

The corresponding expression of signal amplification factor $\eta(\Omega,D,A_0)$ can therefore be obtained using Eq. (2.14) as

$$\eta(\Omega,D,A_0) \approx \eta^{(L)}(\Omega,D)\left[1 - \left(\frac{A_0^2}{4D^2}\right)f^{(ad)}(D)\right] - \left(\frac{A_0^2}{2D^2}\right)\left(\frac{\Omega^2}{\lambda_1^2+\Omega^2}\right)\left[\eta^{(L)}(\Omega,D)fac_1(D) + \left(\frac{\nu}{2D^2}\right)fac_2(D)\right], \quad (4.28)$$

where $\eta^{(L)}(\Omega,D)$ is given in Eq. (4.10). The simplicity of the above expressions [Eqs. (4.27) and (4.28)] lie in the fact that the nonlinear modification of the response requires only the knowledge of the first nontrivial eigenvalue, λ_1, $\langle \phi_0^\dagger|x^2|\phi_0\rangle$ and $\langle \phi_0^\dagger|x|\phi_1\rangle^2$. In accordance with Eq. (2.13) it is clear that the above approximate expression will be valid where

$$\bar{C}_{coh}(\Omega,D,A_0) \geq 0. \quad (4.29)$$

This limits the region in the parameter space (Ω,D,A_0) where we can find out the value of the response directly using the simple formulas [Eqs. (4.27) and (4.28)]. This restriction arises because we have not considered the nonlinear response higher than the leading-order nonlinear response.

C. Comparison with adiabatic theory

In this subsection we compare the approximate expressions [Eq. (4.27)] of leading-order nonlinear response with the corresponding expression obtained from the adiabatic theory. It is known [2] that the adiabatic theory is valid when the frequency of the external signal, Ω is small compared to all other typical frequencies of the system so that the assumption $\Omega t \approx$ const can be made plausible. The asymptotic probability distribution is given by [2,13]

$$p_0(x,t) = Z^{-1}\exp\left[-\frac{V(x)}{D} + \beta x\right], \quad (4.30)$$

where $V(x)$ is given in Eq. (2.2) and the quantities β and Z are

$$\beta = \left(\frac{A_0}{D}\right)\cos\Omega t, \quad (4.31)$$

$$Z = \int_{-\infty}^{\infty} dx \exp\left[-\frac{V(x)}{D} + \beta x\right]$$
$$= Z_0\left[\mathcal{D}_{-1/2}(-1/\sqrt{2D}) + \beta^2\left(\frac{\sqrt{2D}}{4}\right)\mathcal{D}_{-3/2}(-1/\sqrt{2D}) + \frac{1}{2!}\beta^4\left(\frac{\sqrt{2D}}{4}\right)^2\mathcal{D}_{-5/2}(-1/\sqrt{2D}) + O(\beta^6)\right], \quad (4.32)$$

$$Z_0 = e^{(1/8D)}(2D)^{1/4}\sqrt{\pi}. \quad (4.33)$$

The quantities $\mathcal{D}_{-n}$ in Eq. (4.32) denote parabolic cylinder functions. We attempt to obtain a closed analytical form of $\bar{C}_{coh}(\Omega, D, A_0)$ which will be correct up to $O(A_0^4)$. Therefore we first try to evaluate $\langle X(t)\rangle_{asy}$ from Eq. (4.30) which will be correct up to $O(A_0^3)$. From Eq. (4.32) it is easy to see that

$$\langle X(t)\rangle_{asy} = 2\left(\frac{\sqrt{2D}}{4}\right)\left(\frac{\mathcal{D}_{-3/2}(-1/\sqrt{2D})}{\mathcal{D}_{-1/2}(-1/\sqrt{2D})}\right) \times\beta\left[1 + \left(\frac{\sqrt{2D}}{4}\right)\beta^2\left(\frac{\mathcal{D}_{-5/2}(-1/\sqrt{2D})}{\mathcal{D}_{-3/2}(-1/\sqrt{2D})} - \frac{\mathcal{D}_{-3/2}(-1/\sqrt{2D})}{\mathcal{D}_{-1/2}(-1/\sqrt{2D})}\right) + O(\beta^4)\right]. \quad (4.34)$$

It is however convenient for our purpose to express $\langle X(t)\rangle_{asy}$ in terms of $\langle\phi_0^\dagger|x^2|\phi_0\rangle$. From Eq. (4.32) one derives

$$\langle\phi_0^\dagger|x^2|\phi_0\rangle = 2\left(\frac{\sqrt{2D}}{4}\right)\left(\frac{\mathcal{D}_{-3/2}(-1/\sqrt{2D})}{\mathcal{D}_{-1/2}(-1/\sqrt{2D})}\right), \quad (4.35)$$

$$\langle\phi_0^\dagger|x^4|\phi_0\rangle = \left(\frac{4!}{2!}\right)\left(\frac{\sqrt{2D}}{4}\right)^2\left(\frac{\mathcal{D}_{-5/2}(-1/\sqrt{2D})}{\mathcal{D}_{-1/2}(-1/\sqrt{2D})}\right). \quad (4.36)$$

Using Eqs. (4.35) and (4.36) we thus have

$$\left(\frac{\sqrt{2D}}{4}\right)\left(\frac{\mathcal{D}_{-5/2}(-1/\sqrt{2D})}{\mathcal{D}_{-3/2}(-1/\sqrt{2D})}\right) = \left(\frac{1}{6}\right)\frac{\langle\phi_0^\dagger|x^4|\phi_0\rangle}{\langle\phi_0^\dagger|x^2|\phi_0\rangle} = \left(\frac{1}{6}\right)\left(1 + \frac{D}{\langle\phi_0^\dagger|x^2|\phi_0\rangle}\right), \quad (4.37)$$

where in the last equality we have used Eq. (B10). Inserting the results Eqs. (4.35) and (4.37) into the expression of $\langle X(t)\rangle_{asy}$, we obtain

$$\langle X(t)\rangle_{asy} = \langle\phi_0^\dagger|x^2|\phi_0\rangle\beta\left[1 - \frac{1}{6}f^{(ad)}\beta^2 + O(\beta^4)\right], \quad (4.38)$$

where $f^{(ad)}$ is given in Eq. (4.26). Employing this expression of asymptotic mean value $\langle X(t)\rangle_{asy}$ in Eqs. (2.10)–(2.12) we obtain the adiabatic coherent response as

$$\bar{C}_{coh}^{(ad)}(\Omega, D, A_0) \equiv \bar{C}_{coh}^{(ad)}(D, A_0) = \bar{C}_{coh}^{(L)(ad)}(D, A_0)\left\{1 - \left(\frac{A_0^2}{4D^2}\right)f^{(ad)}(D) + O\left[\left(\frac{A_0}{D}\right)^4\right]\right\}, \quad (4.39)$$

where the quantity $\bar{C}_{coh}^{(L)(ad)}(D, A_0)$ is given by

$$\bar{C}_{coh}^{(L)(ad)}(D, A_0) = \left(\frac{A_0^2}{2D^2}\right)\langle\phi_0^\dagger|x^2|\phi_0\rangle^2. \quad (4.40)$$

It is thus observed that in the adiabatic limit when the frequency of the external signal is smaller than all other typical frequencies (inverse of the eigenvalues of the unperturbed Fokker Planck operator) the coherent response is independent of the external frequency. In the low noise regime, in the limit $\Omega \to 0$ the expressions Eqs. (4.8), (4.9), and (4.27) of linear and nonlinear coherent responses correctly go over to the corresponding adiabatic expressions Eqs. (4.40) and (4.39), respectively. Frequency dependence of the coherent response manifests in a theory which is beyond the adiabatic theory.

D. Comparison with numerical results

With typical values of $D=0.2$, $\Omega=0.1$ the approximate expression [Eq. (4.27)] of $\bar{C}_{coh}(\Omega, D, A_0)$ is plotted against A_0^2 (solid curve) and compared with the numerical values (open circles) [2,4] obtained by directly solving the Langevin equation Eqs. (2.1)–(2.3) in Fig. 1. Initial few values are extracted from the numerical data of the signal amplification factor, η [2] and translated to $\bar{C}_{coh}$ via Eq. (2.14). It agrees reasonably well with initial few values. The curve shows that $\bar{C}_{coh} < \bar{C}_{coh}^{(L)}$ almost always. This feature agrees with the numerical results. However, the curve for $\bar{C}_{coh}(\Omega=0.1, D=0.2, A_0)$ after initial increase with A_0^2 shows gradual fall with A_0^2 in the restricted region of validity determined by Eq. (4.29). This happens because of the following reason. The coefficient α in Eq. (4.9) is always positive and $\bar{C}_{coh}^{(L)}(\Omega, D, A_0)$ in Eq. (4.8) is proportional to A_0^2 while $\bar{C}_{coh}^{(NL)}$ which is proportional to A_0^4 is negative. Thus $\bar{C}_{coh}(\Omega=0.1, D=0.2, A_0)$ in Eq. (4.27) must decrease after some value of A_0^2. The direct numerical evaluation [4] shows that generally $\bar{C}_{coh}(\Omega, D, A_0)$ does not fall but tends to saturate very fast as A_0^2 increases. This could happen due to the contribution of higher-order terms in the perturbation series while we have considered the terms up to the leading-order nonlinear response [Eq. (4.27)]. Therefore although the leading-order nonlinear response Eq. (4.27) takes into account the deviation from the linear-response contribution quite substantially in the restricted regime [Eq. (4.29)], higher order than the leading-order nonlinear response is essential to arrest the inevitable downward fall.

In order to observe the dependence of the response on the noise strength for fixed amplitude of the external signal, the approximate expression [Eq. (4.28)] of $\eta(\Omega, D, A_0)$ is plotted

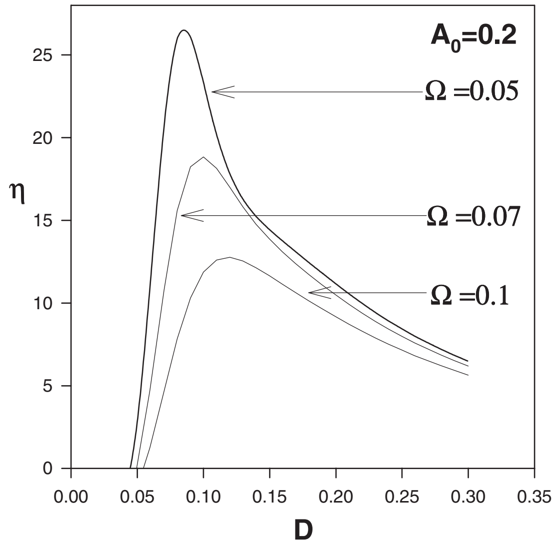

FIG. 3. η vs D with $A_0=0.2$ and $\Omega=0.05,0.07,0.1$.

against D for typical values of $A_0=0.1$, $\Omega=0.1$ (solid curve) and compared with the numerical values (dotted curve) [2] obtained using the matrix-continued-fraction (MCF) technique [10] in Fig. 2. This curve more or less agrees with the numerical results. The curve with $A_0=0.3$ although more or less agrees with the numerical data for comparatively higher values of D but fails to reproduce the numerical results for comparatively lower values of D. The peak value of $\eta(A_0=0.3)$ shows greater than the maximum of $\eta^{(L)}$ while numerical data exhibits lesser value. For comparatively high values of D, η is found to be always less than $\eta^{(L)}$. This feature agrees with the numerical result. For comparatively low D region the numerical result shows $\eta > \eta^{(L)}$ and shifting of the peak position of η toward the left of that of $\eta^{(L)}$. Although this feature develops in the leading-order nonlinear response but it yields very large value of the signal amplification factor.

The nature of the natural appearance of D in the form of $(\frac{A_0}{D})$ in the perturbation expansion of the response Eqs. (4.8), (4.9), (4.27), and (4.28) with respect to the amplitude shows that these expressions would yield comparatively better values for high values of the noise strength with fixed value of A_0 because the series would converge faster. The Fig. 1 demonstrates that with fixed value of the noise strength, D the expression of $\bar{C}_{coh}$ fails to reproduce the numerical results for high values of A_0. As powers of $(\frac{A_0}{D})^2$ appears naturally in the perturbation expansion, this indicates for fixed value of the amplitude the leading-order nonlinear response would fail to reproduce the numerical values for comparatively low values of D because the convergence would be poorer and consequently higher-order terms are necessary.

In order to observe the dependence of the response on the frequency for fixed amplitude of the external signal, the signal amplification factor, $\eta(\Omega,D,A_0)$ for different frequencies are calculated from the expression Eq. (4.28) for a fixed value of the amplitude $A_0=0.2$ and are plotted against the noise strength in Fig. 3. This figure shows the enhancement of the amplification factor and shifting of the peak positions toward low noise values as the frequency decreases. These features are in accordance with the numerical results in [2].

As stated before, the Ω dependence of the coherent response is a manifestation of the perturbation theory, which is absent in the adiabatic theory. In adiabatic theory [2] the time-dependent probability distribution is exponentially concentrated in the right or in the left well. Therefore the mean value performs an oscillation between the positions of the two minima. The finite noise disturbs the coherency of the oscillation and consequently the amplitude of the coherent oscillation of the mean value decreases, i.e., it reduces the signal amplification factor. Figure 3 shows that the increase in the frequency of the external signal influences the system similarly. Roughly, coherency is lost because of mismatch of the frequency of the external signal and hopping frequency λ_1 between the two wells.

It is observed from Fig. 3 that the values of η increases as the frequency decreases. It is also shown in Sec. IV C that $\bar{C}_{coh} \to \bar{C}_{coh}^{(ad)}$ and therefore $\eta \to \eta^{(ad)}$ as $\Omega \to 0$. Thus we expect

$$\eta(\Omega,D,A_0) < \eta^{(ad)}(D,A_0) = \eta^{(L)(ad)}\left\{1 - \frac{A_0^2}{4D^2}f^{(ad)} + O\left[\left(\frac{A_0}{D}\right)^4\right]\right\},$$

$$\eta^{(L)(ad)} = \left(\frac{\langle \phi_0^\dagger | x^2 | \phi_0 \rangle}{D}\right)^2, \qquad (4.41)$$

where the region in the parameter space should be restricted by the physical condition $\eta^{(ad)} \geq 0$. As has been argued before, since successively higher powers of $(\frac{A_0}{D})^2$ appear in the expression, the truncated expression which is correct up to $O[(\frac{1}{D^2})(\frac{A_0}{D})^2]$ would provide better estimate of $\eta^{(ad)}$ for comparatively higher values of D.

V. CONCLUSIONS

In this paper we consider an overdamped bistable system driven by a Gaussian white noise and a monochromatic periodic force. We calculate the response in terms of the one-time correlation function (defined in the text earlier) of the bistable system at the input frequency of applied monochromatic force. With small amplitude of the driving force the perturbation theory is developed systematically as a power series of the amplitude of the periodic force to calculate the coherent component of the response. The first term of this series is called the linear coherent response. The next higher-order term in the amplitude is defined as the leading-order nonlinear response. In Sec. III we derive these quantities in terms of the eigenvalues $\{\lambda_l\}$ and the eigenfunctions $\{\phi_l(x)\}$ of the unperturbed Fokker-Planck operator $\hat{L}_0$. These expressions constitute several summations involving the full set of the eigenvalues and the eigenfunctions of $\hat{L}_0$. In Sec. IV it is shown that taking into account of the leading-order nonlinear response an approximate analytical expressions of the coherent responses can be obtained without detailed knowledge of $\{\lambda_l\}$ and $\{\phi_l(x)\}$ with $l \geq 2$ in low-frequency low noise regime where the conditions, Eqs. (4.1)–(4.3) are satisfied. Expression (4.27) and (4.28) involve only the first nontrivial eigenvalue λ_1 which relates to the rate of escape from the potential well, the corresponding eigenfunction $\phi_1(x)$ and the lowest eigenfunction $\phi_0(x)$ corresponding to zero eigenvalue.

This analytical formula is derived with the help of several results that have been proved in the Appendixes A–G.

In Sec. IV C the expression of the leading-order nonlinear response in adiabatic regime has been derived exactly. As the perturbation expansion with respect to the amplitude of the periodic signal brings out the frequency dependence of the coherent response it is beyond the adiabatic theory. It is shown that our expressions Eqs. (4.8)–(4.10), (4.27), and (4.28) correctly go over to the corresponding results of the adiabatic theory as the frequency, $\Omega \to 0$.

The A_0, D and Ω dependence of the derived expressions (4.27) and (4.28) of the coherent response/the signal amplification factor have been illustrated and compared with the numerical results in Figs. 1–3 respectively. It is generally observed that these approximate expressions more or less agree with the numerical results for low values of A_0 and comparatively high values of D. These analytical expressions are derived in the parameter regime where Eqs. (4.1)–(4.3) are valid. This is the first restriction imposed on the regime of the parameter space (Ω, D, A_0). The linear coherent response appears as first nontrivial term in the perturbation theory and it is correct up to $O[(\frac{A_0}{D})^2]$. Figures 1 and 2 show its limitation in regards to its range of applicability. We have calculated response up to the second nontrivial term which is correct up to $O[(\frac{A_0}{D})^4]$. This imposes the second restriction to the parameter regime. Consequence of the termination of the perturbation series manifests in the downward fall of $\bar{C}_{coh}$ with high values of A_0^2 with fixed $D(D=0.2)$ in Fig. 1 and high values of η for low values of D with fixed $A_0(A_0=0.3)$ in Fig. 2. Therefore even if these expressions, Eqs. (4.27) and (4.28) are correct up to $O[(\frac{A_0}{D})^4]$, the calculation shows that they do not reproduce the numerical results for comparatively high value of A_0 and comparatively low values of D. It is argued that as successive powers of $(A_0/D)^2$ appear naturally in the perturbation expansion, the convergence of the perturbation series will be poorer for comparatively lower values of D with fixed A_0 and for high values of A_0 with fixed value of D. Hence the calculations of higher-order terms in the perturbation series is in order, which would be taken up for future communications. Besides, more numerical results by directly solving the Langevin equation for either coherent response or signal amplification factor covering a wide range of parameter space are necessary in order to make a definite statement of the range of validity of these simple expressions. The simplicity and usefulness of these expressions lie in the fact that only the first nontrivial eigenvalue, λ_1, $\langle\phi_0^\dagger|x^2|\phi_0\rangle$ and $\langle\phi_0^\dagger|x|\phi_1\rangle^2$ are necessary to evaluate these responses and once the upper and lower limits of the range of validity are determined through numerical check these expressions would provide a first hand quick estimate of the responses.

APPENDIX A: PROOF: $2Da_{lr}=-\langle\phi_l^\dagger|V'(x)|\phi_r\rangle-(\lambda_l-\lambda_r)\langle\phi_l^\dagger|x|\phi_r\rangle$

The eigenvalue equation for the Fokker-Planck operator $\hat{L}_0$ in Eqs. (2.5) and (3.1) is explicitly written as

$$\frac{\partial}{\partial x}[V'(x)\phi_r(x)] + D\frac{\partial^2}{\partial x^2}\phi_r(x) = -\lambda_r\phi_r(x). \tag{A1}$$

Multiplying Eq. (A1) by $x\phi_l^\dagger(x)$ and integrating by parts with the boundary conditions that $\phi_l^\dagger(x)$, $\phi_r(x)$ and their derivatives vanish very fast as $|x|\to\infty$, we obtain

$$\begin{aligned}&-\int dx\phi_l^\dagger(x)V'(x)\phi_r(x) - \int dx\, x\frac{\partial\phi_l^\dagger(x)}{\partial x}V'(x)\phi_r(x)\\ &\quad - D\int dx\, x\frac{\partial\phi_l^\dagger(x)}{\partial x}\frac{\partial\phi_r(x)}{\partial x} - D\int dx\phi_l^\dagger(x)\frac{\partial\phi_r(x)}{\partial x}\\ &= -\lambda_r\int dx\, x\phi_l^\dagger(x)\phi_r(x).\end{aligned} \tag{A2}$$

The eigenvalue equation for the adjoint Fokker-Planck operator $\hat{L}_0^\dagger$ in Eq. (3.3) is

$$-V'(x)\frac{\partial\phi_l^\dagger(x)}{\partial x} + D\frac{\partial^2\phi_l^\dagger(x)}{\partial x^2} = -\lambda_l\phi_l^\dagger(x). \tag{A3}$$

Multiplying Eq. (A3) by $x\phi_r(x)$ and integrating by parts we obtain

$$\begin{aligned}&-\int dx\, x\frac{\partial\phi_l^\dagger(x)}{\partial x}V'(x)\phi_r(x) - D\int dx\, x\frac{\partial\phi_l^\dagger(x)}{\partial x}\frac{\partial\phi_r(x)}{\partial x}\\ &\quad + D\int dx\phi_l^\dagger(x)\frac{\partial\phi_r(x)}{\partial x} = -\lambda_l\int dx\, x\phi_l^\dagger(x)\phi_r(x).\end{aligned} \tag{A4}$$

Subtracting Eq. (A2) from Eq. (A4) one obtains

$$2Da_{lr} = -\langle\phi_l^\dagger|V'(x)|\phi_r\rangle - (\lambda_l-\lambda_r)\langle\phi_l^\dagger|x|\phi_r\rangle, \tag{A5}$$

where a_{lr} is defined as in Eq. (3.30).

Special cases:

(a) Proof: $Da_{m0}=-\lambda_m\langle\phi_m^\dagger|x|\phi_0\rangle, \ \forall \ m$.

Interchanging the dummy subscripts l and r in Eq. (A5) and subtracting the resulting equation from the original one, we get

$$D(a_{lr}-a_{rl}) = -(\lambda_l-\lambda_r)\langle\phi_l^\dagger|x|\phi_r\rangle. \tag{A6}$$

Substituting $l=m$, $r=0$ and noting that $a_{0m}=\int dx\frac{\partial}{\partial x}\phi_m(x)=0, \ \forall \ m$, one has

$$Da_{m0} = -\lambda_m\langle\phi_m^\dagger|x|\phi_0\rangle. \tag{A7}$$

(b) Proof: $\langle\phi_m^\dagger|x^3|\phi_0\rangle=(\lambda_m+1)\langle\phi_m^\dagger|x|\phi_0\rangle, \ m\geq 1$.

Interchanging the dummy subscripts l and r in Eq. (A5), replacing l by m and taking $r=0$ in the resulting equation, we have

$$\langle\phi_0^\dagger|V'(x)|\phi_m\rangle = \lambda_m\langle\phi_0^\dagger|x|\phi_m\rangle. \tag{A8}$$

Taking the potential $V(x)=-\frac{1}{2}x^2+\frac{1}{4}x^4$, we immediately obtain from Eq. (A8)

$$\langle\phi_0^\dagger|x^3|\phi_m\rangle=(\lambda_m+1)\langle\phi_0^\dagger|x|\phi_m\rangle,\ \ m\geq 1. \tag{A9}$$

APPENDIX B: PROOF: $2Dk\langle\phi_l^\dagger|x^{k-1}\frac{\partial}{\partial x}|\phi_r\rangle+Dk(k-1)\langle\phi_l^\dagger|x^{k-2}|\phi_r\rangle=-k\langle\phi_l^\dagger|x^{k-1}V'(x)|\phi_r\rangle-(\lambda_l-\lambda_r)\langle\phi_l^\dagger|x^k|\phi_r\rangle,\ \ k\geq 2$

In this appendix we generalize the proposition of Appendix A. Here we prove a relation involving $\langle\phi_l^\dagger|x^k|\phi_r\rangle$, $k\geq 2$, while in Appendix A we have proved a relation involving $\langle\phi_l^\dagger|x|\phi_r\rangle$; it is not allowed to put $k=1$ in the proposition of Appendix B.

Multiplying Eq. (A1) by $x^k\phi_l^\dagger(x)(k\geq 2)$ and integrating by parts with the boundary conditions that $\phi_l^\dagger(x)$, $\phi_r(x)$ and their derivatives vanish very fast as $|x|\to\infty$, we obtain

$$-k\int dx\phi_l^\dagger(x)x^{k-1}V'(x)\phi_r(x)-\int dx\,x^k\frac{\partial\phi_l^\dagger(x)}{\partial x}V'(x)\phi_r(x)-D\int dx\,x^k\frac{\partial\phi_l^\dagger(x)}{\partial x}\frac{\partial\phi_r(x)}{\partial x}-Dk\int dx\phi_l^\dagger(x)x^{k-1}\frac{\partial\phi_r(x)}{\partial x}=-\lambda_r\int dx\phi_l^\dagger(x)x^k\phi_r(x). \tag{B1}$$

Multiplying Eq. (A3) by $x^k\phi_r(x)(k\geq 2)$ and integrating by parts we obtain

$$Dk(k-1)\int dx\phi_l^\dagger(x)x^{k-2}\phi_r(x)-\int dx\,x^k\frac{\partial\phi_l^\dagger(x)}{\partial x}V'(x)\phi_r(x)-D\int dx\,x^k\frac{\partial\phi_l^\dagger(x)}{\partial x}\frac{\partial\phi_r(x)}{\partial x}+Dk\int dx\phi_l^\dagger(x)x^{k-1}\frac{\partial\phi_r(x)}{\partial x}=-\lambda_l\int dx\phi_l^\dagger(x)x^k\phi_r(x). \tag{B2}$$

Subtracting Eq. (B1) from Eq. (B2) one obtains

$$2Dk\langle\phi_l^\dagger|x^{k-1}\frac{\partial}{\partial x}|\phi_r\rangle+Dk(k-1)\langle\phi_l^\dagger|x^{k-2}|\phi_r\rangle=-k\langle\phi_l^\dagger|x^{k-1}V'(x)|\phi_r\rangle-(\lambda_l-\lambda_r)\langle\phi_l^\dagger|x^k|\phi_r\rangle,\ \ k\geq 2. \tag{B3}$$

Special cases:

(1) Interchanging the dummy subscripts l and r for convenience and taking $k=2$ in Eq. (B3) we get

$$4D\langle\phi_r^\dagger|x\frac{\partial}{\partial x}|\phi_l\rangle+2D\delta_{l,r}=-2\langle\phi_r^\dagger|xV'(x)|\phi_l\rangle-(\lambda_r-\lambda_l)\langle\phi_r^\dagger|x^2|\phi_l\rangle. \tag{B4}$$

(1a) Proof: $\langle\phi_r^\dagger|x\frac{\partial}{\partial x}|\phi_0\rangle=-(\frac{\lambda_r}{2D})\langle\phi_r^\dagger|x^2|\phi_0\rangle$, $r\geq 2$.

Taking $l=0$ and $r\neq 0$, Eq. (B4) takes the form

$$4D\langle\phi_r^\dagger|x\frac{\partial}{\partial x}|\phi_0\rangle=-2\langle\phi_r^\dagger|xV'(x)|\phi_0\rangle-\lambda_r\langle\phi_r^\dagger|x^2|\phi_0\rangle. \tag{B5}$$

The lowest eigenfunction $\phi_0(x)$, corresponding to $\lambda_0=0$ in Eq. (A1) satisfies

$$V'(x)\phi_0(x)+D\phi_0'(x)=0. \tag{B6}$$

Employing Eq. (B6) the above equation reduces to

$$\langle\phi_r^\dagger|x\frac{\partial}{\partial x}|\phi_0\rangle=-\left(\frac{\lambda_r}{2D}\right)\langle\phi_r^\dagger|x^2|\phi_0\rangle;\ \ r\geq 2. \tag{B7}$$

(1b) Proof: $\langle\phi_0^\dagger|x^4|\phi_0\rangle=D+\langle\phi_0^\dagger|x^2|\phi_0\rangle$.

Taking $l=0$ and $r=0$, Eq. (B4) takes the form

$$2D\langle\phi_0^\dagger|x\frac{\partial}{\partial x}|\phi_0\rangle+D=-\langle\phi_0^\dagger|xV'(x)|\phi_0\rangle. \tag{B8}$$

Employing Eq. (B6) in Eq. (B8) we get

$$\langle\phi_0^\dagger|xV'(x)|\phi_0\rangle=D. \tag{B9}$$

With $V(x)$ given in Eq. (2.2) one readily arrives at

$$\langle\phi_0^\dagger|x^4|\phi_0\rangle=D+\langle\phi_0^\dagger|x^2|\phi_0\rangle. \tag{B10}$$

(2) Proof: $\langle\phi_m^\dagger|x^2\frac{\partial}{\partial x}|\phi_0\rangle=-2\langle\phi_m^\dagger|x|\phi_0\rangle-(\frac{\lambda_m}{3D})\langle\phi_m^\dagger|x^3|\phi_0\rangle$, $m\geq 1$.

Replacing the dummy index l by m for convenience and taking $r=0$, $k=3$, Eq. (B3) takes the form

$$6D\langle\phi_m^\dagger|x^2\frac{\partial}{\partial x}|\phi_0\rangle+6D\langle\phi_m^\dagger|x|\phi_0\rangle=-3\langle\phi_m^\dagger|x^2V'(x)|\phi_0\rangle-\lambda_m\langle\phi_m^\dagger|x^3|\phi_0\rangle. \tag{B11}$$

Employing Eq. (B6) the above equation boils down to

$$\langle\phi_m^\dagger|x^2\frac{\partial}{\partial x}|\phi_0\rangle=-2\langle\phi_m^\dagger|x|\phi_0\rangle-\left(\frac{\lambda_m}{3D}\right)\langle\phi_m^\dagger|x^3|\phi_0\rangle,\ \ m\geq 1. \tag{B12}$$

APPENDIX C: PROOF: $Da_{rl}=\lambda_r\langle\phi_l^\dagger|\eta_r\rangle$

The quantities $\{a_{lr}\}$ which are defined in Eq. (3.30) involve the sets of eigenfunctions $\{\phi_l^\dagger(x)\}$ and $\{\phi_r(x)\}$ of the unperturbed operators $\hat{L}_0^\dagger$ and $\hat{L}_0$. In this appendix we will show that it is possible to recast these quantities in a different form [see Eq. (C5) below]. It will be shown in Appendixes F and G that this form will be very useful to evaluate the summations involving C_1 in Eqs. (4.13) and (4.14) in approximately closed form.

Integrating Eq. (A1) from $-\infty$ to x and assuming $\phi_r(x)$ and its derivative vanish very fast as $|x|\to\infty$, we obtain

$$[V'(x)\phi_r(x)]+D\frac{\partial}{\partial x}\phi_r(x)=-\lambda_r\eta_r(x), \tag{C1}$$

where

$$\eta_r(x) = \int_{-\infty}^{x} dy\,\phi_r(y). \tag{C2}$$

Multiplying Eq. (C1) by $\phi_l^\dagger(x)$ and integrating from $-\infty$ to ∞ we have

$$\langle \phi_l^\dagger | V'(x) | \phi_r \rangle + Da_{lr} = -\lambda_r \langle \phi_l^\dagger | \eta_r \rangle. \tag{C3}$$

Eliminating $\langle \phi_l^\dagger | V'(x) | \phi_r \rangle$ from Eqs. (C3) and (A5) we get

$$Da_{lr} = -(\lambda_l - \lambda_r)\langle \phi_l^\dagger | x | \phi_r \rangle + \lambda_r \langle \phi_l^\dagger | \eta_r \rangle. \tag{C4}$$

Eliminating $(\lambda_l - \lambda_r)\langle \phi_l^\dagger | x | \phi_r \rangle$ from Eqs. (C4) and (A6) we finally obtain

$$Da_{rl} = \lambda_r \langle \phi_l^\dagger | \eta_r \rangle. \tag{C5}$$

APPENDIX D: EQUATIONS FOR $\eta_r(x)$ AND $\xi_r(x)$

In this appendix we will identify the quantities $\{\eta_r(x)\}$, introduced in Appendix C, as the eigenfunctions of the adjoint Fokker-Planck operator with the inverted (upside-down) potential $[V(x) \to -V(x)]$ and obtain the normalizing condition [see Eq. (D8) below] satisfied by them. This condition will be employed to obtain one important relation in Appendix E.

With $\eta_r(x)$ defined in Eq. (C2) the eigenvalue equation for $\eta_r(x)$ can be read from Eq. (C1)

$$V'(x)\frac{\partial}{\partial x}\eta_r(x) + D\frac{\partial^2}{\partial x^2}\eta_r(x) = \hat{L}_0^\dagger(x;-V)\eta_r(x) = -\lambda_r \eta_r(x), \tag{D1}$$

where in the definition [Eq. (A3)] of $\hat{L}_0^\dagger$ we replace $V(x)$ by $-V(x)$.

Writing

$$\eta_r(x) = \phi_0(x)\xi_r(x) \tag{D2}$$

in Eq. (D1) we arrive at the equation of $\xi_r(x)$:

$$\frac{\partial}{\partial x}[-V'(x)\xi_r(x)] + D\frac{\partial^2}{\partial x^2}\xi_r(x) = \hat{L}_0(x;-V)\xi_r(x) = -\lambda_r \xi_r(x). \tag{D3}$$

Equation (D3) is an eigenvalue equation for the inverted (upside-down) potential $[V(x) \to -V(x)]$. The two reflecting boundary conditions at $x \to -\infty$ and $x \to \infty$ of the original problem Eq. (3.1) are transformed to two absorbing boundary conditions. Therefore the subscript r of the eigenvalues $\{\lambda_r\}$ in Eq. (D3) runs from 1 to ∞ [10]. From the definition of $\eta_r(x)$ in Eq. (C2) it is immediately clear that $\eta_r(-\infty)=0$. Further, from the normalization condition of $\{\phi_r(x)\}$ in Eq. (3.2) with $\phi_0^\dagger=1$ we have $\eta_r(\infty)=0$.

Next, we are going to derive the normalization condition for $\{\eta_l(x)\}$. Taking derivative of Eq. (D2) with respect to x we get

$$-\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)\xi_r(x) + \phi_r^\dagger(x) - \xi_r'(x) = 0. \tag{D4}$$

Again, using Eq. (B6) the equation for $\eta_l(x)$ in Eq. (C1) rewrites as

$$-\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)\phi_l(x) + \phi_l'(x) = -\left(\frac{\lambda_l}{D}\right)\eta_l(x). \tag{D5}$$

Multiplying Eq. (D4) by $\phi_l(x)$ and Eq. (D5) by $\xi_r(x)$ and subtracting the resulting equations we obtain

$$\frac{\partial}{\partial x}[\xi_r(x)\phi_l(x)] - \phi_r^\dagger(x)\phi_l(x) = -\left(\frac{\lambda_l}{D}\right)\xi_r(x)\eta_l(x). \tag{D6}$$

Integrating Eq. (D6) from $x=-\infty$ to $x=\infty$, we have

$$[\xi_r(x)\phi_l(x)]_{x=-\infty}^{x=\infty} - \langle \phi_r^\dagger | \phi_l \rangle = -\left(\frac{\lambda_l}{D}\right)\int_{-\infty}^{\infty} dx\,\xi_r(x)\eta_l(x). \tag{D7}$$

We note that $\xi_r(x)\phi_l(x) = \eta_r(x)\phi_l^\dagger(x)$ and since $\eta_r(x)$, $\phi_l^\dagger(x) \to 0$, as $x \to \pm\infty$; r, $l \neq 0$, we finally arrive at the normalization condition for $\{\eta_l(x)\}$:

$$\left(\frac{\lambda_l}{D}\right)\int_{-\infty}^{\infty} dx\,\xi_r(x)\eta_l(x) = \langle \phi_r^\dagger | \phi_l \rangle = \delta_{r,l}. \tag{D8}$$

APPENDIX E: PROOF: $\sum_{l=1}^{\infty}\frac{a_{lr}^2}{\lambda_l}=\frac{1}{D}$

The quantity $\eta_l(x)$ satisfies Eq. (D1) and the normalization condition satisfied by $\{\eta_l(x)\}$ is given by Eq. (D8). We assume that $\{\eta_l(x)\}_{l=1}^{\infty}$ form a complete set so that any suitable vector $|f\rangle$ is expandable in the basis $\{|\eta_l\rangle\}_{l=1}^{\infty}$:

$$|f\rangle = \sum_{l=1}^{\infty} C_l |\eta_l\rangle, \tag{E1}$$

where Dirac's bra-ket notation has been used. Taking inner product of Eq. (E1) with $\langle \xi_l|$ and using the normalization condition Eq. (D8), the coefficient C_l is obtained as

$$C_l = \left(\frac{\lambda_l}{D}\right)\langle \xi_l | f \rangle. \tag{E2}$$

Substituting Eq. (E2) into Eq. (E1) we obtain

$$\sum_{l=1}^{\infty}\left(\frac{\lambda_l}{D}\right)|\eta_l\rangle\langle \xi_l| = 1. \tag{E3}$$

We now take inner product of the operator equation Eq. (E3) with $\langle \phi_r^\dagger|$ from left and $|\phi_r\rangle$ from right to obtain

$$1 = \sum_{l=1}^{\infty}\left(\frac{\lambda_l}{D}\right)\langle \phi_r^\dagger | \eta_l \rangle \langle \xi_l | \phi_r \rangle = \sum_{l=1}^{\infty}\left(\frac{\lambda_l}{D}\right)\langle \phi_r^\dagger | \eta_l \rangle \left\langle \frac{\eta_l}{\phi_0} \Big| \phi_r \right\rangle$$
$$= \sum_{l=1}^{\infty}\left(\frac{\lambda_l}{D}\right)\langle \phi_r^\dagger | \eta_l \rangle^2, \tag{E4}$$

where the definition [Eq. (D2)] of ξ_l has been employed in the second equality. Employing the alternate expression [Eq. (C5)] of a_{lr}, Eq. (E4) takes the form:

$$\sum_{l=1}^{\infty}\frac{a_{lr}^2}{\lambda_l} = \frac{1}{D}. \tag{E5}$$

In particular, taking $r=0$ in Eq. (E5) and using the result Eq. (A7) we obtain

$$\sum_{l=1}^{\infty} \lambda_l \langle \phi_l^{\dagger}|x|\phi_0\rangle^2 = D. \tag{E6}$$

APPENDIX F: PROOF: $\sum_{m=1}^{\infty}\frac{\langle\phi_0^{\dagger}|x|\phi_m\rangle}{\lambda_m}f_m \approx \left(\frac{\langle\phi_0^{\dagger}|x|\phi_1\rangle}{D^2}\right)\left[\frac{1}{2}(\lambda_1+1)-\langle\phi_0^{\dagger}|x^2|\phi_0\rangle\right]$

In this appendix we prove a relation which will be utilized to evaluate the summation involving C_1 in Eq. (4.14) in closed form.

Let us first note that

$$\sum_{m=1}^{\infty} \frac{\langle\phi_0^{\dagger}|x|\phi_m\rangle}{\lambda_m} a_{mr} = \left(\frac{1}{D}\right)\sum_{m=1}^{\infty}\langle\phi_0^{\dagger}|x|\phi_m\rangle\langle\phi_r^{\dagger}|\eta_m\rangle. \tag{F1}$$

where use has been made of the definition [Eq. (C5)] of a_{mr}. Using the closure relation

$$\sum_{r=0}^{\infty}\phi_r(x)\phi_r^{\dagger}(x') = \delta(x-x') \tag{F2}$$

and noting that $\langle\phi_0^{\dagger}|x|\phi_0\rangle=0$, the summation Eq. (F1) is rewritten as

$$\sum_{m=1}^{\infty} \frac{\langle\phi_0^{\dagger}|x|\phi_m\rangle}{\lambda_m} a_{mr} = \left(\frac{1}{D}\right)\int_{-\infty}^{\infty} dy\,\psi(y)\phi_r^{\dagger}(y), \tag{F3}$$

where the function $\psi(y)$ is defined as

$$\psi(y) = \int_{-\infty}^{y} dy'\,y'\,\phi_0(y'). \tag{F4}$$

Employing again the definition [Eq. (C5)] of a_{rl} with $l=1$ we deduce

$$\sum_{m=1}^{\infty} \frac{\langle\phi_0^{\dagger}|x|\phi_m\rangle}{\lambda_m} f_m = \left(\frac{1}{D^2}\right)\left[\int_{-\infty}^{\infty} dy\,\psi(y)I_1(y) - \left(\int_{-\infty}^{\infty} dy\,\psi(y)\right)\int_{-\infty}^{\infty} dz\,\phi_1^{\dagger}(z)I_2(z)\right], \tag{F5}$$

where the quantity f_m are defined in Eq. (4.21) and $I_1(y)$ and $I_2(z)$ are given by

$$I_1(y) = \int_{y}^{\infty} dz\,\phi_1^{\dagger}(z),\quad I_2(z) = \int_{-\infty}^{z} dz'\,\phi_0(z'). \tag{F6}$$

Noting

$$\psi(-y) = \psi(y),\quad I_1(-y) = I_1(y),\quad I_2(-z) = 1 - I_2(z), \tag{F7}$$

the integrals in Eq. (F5) are simplified to

$$\begin{aligned}\int_{-\infty}^{\infty} dy\,\psi(y)I_1(y) &= 2\int_0^{\infty} dy\,\psi(y)I_1(y)\\ &= 2\left[\int_0^{\infty} dy\,\psi(y)\int_0^{\infty} dz\,\phi_1^{\dagger}(z) - \int_0^{\infty} dy\,\psi(y)\int_0^{y} dz\,\phi_1^{\dagger}(z)\right],\end{aligned} \tag{F8}$$

$$\begin{aligned}\int_{-\infty}^{\infty} dz\,\phi_1^{\dagger}(z)I_2(z) &= 2\int_0^{\infty} ds\,\phi_1^{\dagger}(s)\int_0^{s} dz'\,\phi_0(z')\\ &= \int_0^{\infty} ds\,\phi_1^{\dagger}(s) - 2\int_0^{\infty} ds\,\phi_1^{\dagger}(s)\int_s^{\infty} dz'\,\phi_0(z').\end{aligned} \tag{F9}$$

Hence the required summation reduces to

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{\langle\phi_0^{\dagger}|x|\phi_m\rangle}{\lambda_m} f_m = \left(\frac{2}{D^2}\right)\Bigg[&-\int_0^{\infty} dy\,\psi(y)\int_0^{y} dz\,\phi_1^{\dagger}(z)\\ &+ 2\left(\int_0^{\infty} dy\,\psi(y)\right)\int_0^{\infty} ds\,\phi_1^{\dagger}(s)\\ &\times\int_s^{\infty} dz'\,\phi_0(z')\Bigg].\end{aligned} \tag{F10}$$

We next evaluate the integrals appearing in Eq. (F10). Noting the fact that

$$\psi(y) = -\int_y^{\infty} ds\,s\,\phi_0(s),\quad y \geq 0, \tag{F11}$$

the integral

$$\int_0^{\infty} dy\,\psi(y) = -\int_0^{\infty} dy\,y^2\phi_0(y) = -\frac{\langle\phi_0^{\dagger}|x^2|\phi_0\rangle}{2}, \tag{F12}$$

where we have used $\frac{d\psi}{dy}=y\phi_0(y)$. We next evaluate the other two integrals approximately in the low noise regime:

$$\begin{aligned}\int_0^{\infty} ds\,\phi_1^{\dagger}(s)\int_s^{\infty} dz'\,\phi_0(z') &= \int_0^{\infty} dz'\,\phi_0(z')\int_0^{z'} ds\,\phi_1^{\dagger}(s)\\ &\approx \frac{1}{2}\langle\phi_0^{\dagger}|x|\phi_1\rangle,\end{aligned} \tag{F13}$$

$$\begin{aligned}\int_0^{\infty} dy\,\psi(y)\int_0^{y} dz\,\phi_1^{\dagger}(z) &\approx \int_0^{\infty} dy\,\psi(y)y\\ &\approx -\frac{1}{4}(1+\lambda_1)\langle\phi_0^{\dagger}|x|\phi_1\rangle,\end{aligned} \tag{F14}$$

where we have used $\phi_1(x)\approx\phi_0(x)$; $x>0$ for low D [7], $\psi(\infty)=0$, Eq. (A9) with $m=1$, and the fact that $\phi_0(y)$, $\psi(y)\to 0$ rapidly as $y\to\infty$ so that the above integrals converge. Employing Eqs. (F12)–(F14) into Eq. (F10) we arrive at the required result

$$\sum_{m=1}^{\infty} \frac{\langle \phi_0^\dagger | x | \phi_m \rangle}{\lambda_m} f_m \approx \left(\frac{\langle \phi_0^\dagger | x | \phi_1 \rangle}{D^2} \right) \left[\frac{1}{2}(\lambda_1 + 1) - \langle \phi_0^\dagger | x^2 | \phi_0 \rangle \right]. \tag{F15}$$

APPENDIX G: PROOF: $f_1 = \sum_{r=2}^{\infty} \frac{a_{1r} a_{r1}}{\lambda_r} \approx \left(\frac{\lambda_1}{D^2}\right) \left[\frac{1}{2}\langle \phi_0^\dagger | x^2 | \phi_0 \rangle - \langle \phi_0^\dagger | x | \phi_1 \rangle^2\right]$

In this appendix we evaluate the quantity, f_1 defined through Eq. (4.21), which appears in Eq. (4.13).

Employing the definition [Eq. (C5)] of a_{lr} the quantity f_1 is reduced to, as has been done before:

$$f_1 = \sum_{r=2}^{\infty} \langle \phi_1^\dagger | \eta_r \rangle \langle \phi_r^\dagger | \eta_1 \rangle = \left(\frac{\lambda_1}{D^2} \right) \left[\int_{-\infty}^{\infty} dy\, \eta_1(y) I_1(y) - \left(\int_{-\infty}^{\infty} dy\, \eta_1(y) \right) \int_{-\infty}^{\infty} dz\, \phi_1^\dagger(z) I_2(z) \right], \tag{G1}$$

where the quantities $I_1(y)$ and $I_2(z)$ are, respectively, given by

$$I_1(y) = \int_y^{\infty} dz\, \phi_1^\dagger(z), \quad I_2(z) = \int_{-\infty}^{z} dz'\, \phi_0(z'). \tag{G2}$$

Noting

$$\eta_1(-y) = \eta_1(y), \quad I_1(-y) = I_1(y), \quad I_2(-z) = 1 - I_2(z), \tag{G3}$$

and observing the similarity between Eqs. (G1)–(G3) and Eqs. (F5)–(F7) one writes immediately

$$f_1 = \left(\frac{2}{D^2} \right) \lambda_1 \left[- \int_0^{\infty} dy\, \eta_1(y) \int_0^{y} dz\, \phi_1^\dagger(z) + 2 \left(\int_0^{\infty} dy\, \eta_1(y) \right) \int_0^{\infty} ds\, \phi_1^\dagger(s) \int_s^{\infty} dz'\, \phi_0(z') \right]. \tag{G4}$$

We next evaluate the integrals appearing in Eq. (G4). Employing Eqs. (C5) and (A7) one obtains

$$\int_0^{\infty} dy\, \eta_1(y) = \left(\frac{1}{2} \right) \int_{-\infty}^{\infty} dy\, \eta_1(y) = - \frac{\langle \phi_0^\dagger | x | \phi_1 \rangle}{2}. \tag{G5}$$

Out of the other two integrals one is already evaluated in Eq. (F13). The other one can also be evaluated approximately in the low noise regime as before:

$$\int_0^{\infty} dy\, \eta_1(y) \int_0^{y} dz\, \phi_1^\dagger(z) \approx \int_0^{\infty} dy\, \eta_1(y) y \approx - \frac{1}{4} \langle \phi_0^\dagger | x^2 | \phi_0 \rangle, \tag{G6}$$

where we have used $\phi_1(x) \approx \phi_0(x)$; $x > 0$ for low D [7], $\frac{d\eta_1}{dy} = \phi_1(y)$, $\langle \phi_0^\dagger | \phi_1 \rangle = 0$, and the fact that $\eta_1(y) \to 0$ rapidly as $y \to \infty$ so that the above integral converges. Employing Eqs. (G5), (G6), and (F13) into Eq. (G4) we arrive at the required result

$$f_1 = \sum_{r=2}^{\infty} \frac{a_{1r} a_{r1}}{\lambda_r} \approx \left(\frac{\lambda_1}{D^2} \right) \left[\frac{1}{2} \langle \phi_0^\dagger | x^2 | \phi_0 \rangle - \langle \phi_0^\dagger | x | \phi_1 \rangle^2 \right]. \tag{G7}$$

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